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Tauberian theorems for the (J, p) summability method

İbrahim Çanak [a,*], Ümit Totur [b]

[a] Department of Mathematics, Ege University, Izmir 35100, Turkey

[b] Department of Mathematics, Adnan Menderes University, Aydin 09010, Turkey

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ABSTRACT

In this paper, we prove Tauberian theorems of slowly oscillating type for the (J, p) summability method by using a result due to Mikhalin [G.A. Mikhalin, Theorem of Tauberian type for (J, pn) summation methods, Ukrain. Mat. Zh. 29 (1977), 763–770. English translation: Ukrain. Math. J. 29 (1977) 564–569].

1. Introduction

Assume that $p = (p_n)$ be a sequence of nonnegative numbers with $p_0 > 0$, such that

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as } n \to \infty,$$

and that

$$p(x) = \sum_{k=0}^{\infty} p_k x^k < \infty \quad \text{for } 0 < x < 1.$$

The weighted means of a sequence (u_n) of real numbers are defined by

$$\sigma_{n,p}^{(1)}(u) := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k$$

for all nonnegative integers n.

If

$$\lim_{n\to\infty} \sigma_{n,p}^{(1)}(u) = s, \tag{1}$$

then we say that (u_n) is summable to s by the weighted mean method determined by $p = (p_n)$ and we write $u_n \to s\ (\overline{N}, p)$.

If $\sum_{k=0}^{\infty} p_k u_k x^k$ is convergent for $0 < x < 1$, and $\frac{1}{p(x)} \sum_{k=0}^{\infty} p_k u_k x^k \to s$ as $x \to 1^-$, we say that (u_n) is summable to s by the power series method (J, p), and we write $u_n \to s\ (J, p)$.

* Corresponding author. Tel.: +90 232 311 5418; fax: +90 232 388 1036.
E-mail addresses: ibrahimcanak@yahoo.com, ibrahim.canak@ege.edu.tr (İ. Çanak), utotur@adu.edu.tr (Ü. Totur).

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The $(\overline{N}, p)$ and (J, p) summability methods are regular if and only if $P_n \to \infty$ as $n \to \infty$. It is well known that $u_n \to s\,(\overline{N}, p)$ implies $u_n \to s\,(J, p)$ (see [1]).

The sequence of the backward differences of $u = (u_n)$ is denoted by $\Delta u = (\Delta u_n)$ and defined by $\Delta u_n = u_n - u_{n-1}$ for $n \geq 1$ and $\Delta u_0 = u_0$. The identity

$$u_n - \sigma_{n,p}^{(1)}(u) = V_{n,p}^{(0)}(\Delta u) \tag{2}$$

where $V_{n,p}^{(0)}(\Delta u) := \frac{1}{P_n}\sum_{k=1}^{n} P_{k-1}\Delta u_k$. The identity (2) is called the weighted Kronecker identity and $(V_{n,p}^{(0)}(\Delta u))$ is called a generator sequence of (u_n).

For any nonnegative integer m, we define $\sigma_{n,p}^{(m)}(u)$ and $V_{n,p}^{(m)}(\Delta u)$ by

$$\sigma_{n,p}^{(m)}(u) = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k \sigma_{k,p}^{(m-1)}(u), & m \geq 1 \\ u_n, & m = 0 \end{cases}$$

and

$$V_{n,p}^{(m)}(\Delta u) = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k V_{k,p}^{(m-1)}(\Delta u), & m \geq 1 \\ V_{n,p}^{(0)}(\Delta u), & m = 0 \end{cases}$$

respectively.

If

$$\lim_{n\to\infty} \sigma_{n,p}^{(m)}(u) = s, \tag{3}$$

then we say that (u_n) is summable to s by the $(\overline{N}_m, p)$ method and we write $u_n \to s\,(\overline{N}_m, p)$. If $\sigma_{n,p}^{(m)}(u) \to s\,(J, p)$, we write $u_n \to s\,(J, p)(\overline{N}_m, p)$ for any nonnegative integer m.

A sequence (u_n) is slowly oscillating [2] if

$$\lim_{\lambda\to 1^+} \limsup_{n\to\infty} \max_{n+1\leq k\leq[\lambda n]} \left|\sum_{j=n+1}^{k} \Delta u_j\right| = 0. \tag{4}$$

For the definitions of $O(1)$ and $o(1)$ we refer the reader to [3, page 149].

In this paper our object is to establish the following Tauberian theorems for (J, p) and $(J, p)(\overline{N}_m, p)$ summability methods by using the concept of the generator sequence.

Theorem 1. *Let (p_n) satisfy the conditions*

$$p_0 > 0, \qquad p_n \geq 0 \quad (n = 1, 2, \ldots), \quad P_n = \sum_{k=0}^{n} p_k \to \infty \quad (n \to \infty), \tag{5}$$

$$1 \leq \frac{P_n}{n} \to 1\ (n \to \infty), \tag{6}$$

$$\left(\frac{1}{P_n}\right) \quad \text{is an absolutely monotone sequence}. \tag{7}$$

Then slow oscillation of (u_n) is a Tauberian condition for (J, p) summability.

Theorem 2. *Let (p_n) satisfy the conditions (5)–(7). Then slow oscillation of (u_n) is a Tauberian condition for $(J, p)(\overline{N}_m, p)$ summability.*

Note that in Theorems 1 and 2, if we get $p_n = 1$ for all nonnegative integer n, we have the Tauberian theorems for Abel and generalized Abel summability methods obtained in [4].

Our proofs are based on the following result due to Mikhalin [5].

Theorem 3. *Let (p_n) satisfy the conditions (5),*

$$1 \leq \frac{P_m}{P_n} \to 1 \quad \text{when } 1 < \frac{m}{n} \to 1\ (n \to \infty), \tag{8}$$

and (7). If $V_{n,p}^{(0)}(\Delta u) \geq -C$ for some $C \geq 0$, then $u_n \to s\,(J, p)$ implies $u_n \to s\,(\overline{N}, p)$.

Note that the condition (6) implies the condition (8).

2. Lemmas

We need the following lemmas to prove our theorems.

Lemma 4. *If (λ_n) is an increasing sequence of positive real numbers and (v_n) is a sequence of real numbers, then*

$$\left|\sum_{k=p}^{q}\lambda_k \Delta v_k\right| \le \lambda_q \max_{p\le r\le q}\left|\sum_{k=r}^{q}\Delta v_k\right|.$$

Proof. We have

$$\sum_{k=p}^{q}\lambda_k(v_k - v_{k-1}) = \sum_{k=p}^{q}\lambda_k\left((v_q - v_{k-1}) - (v_q - v_k)\right) = \sum_{k=p}^{q}\lambda_k(S_k - S_{k-1}),$$

where $S_k = \sum_{j=k}^{q}\Delta v_j$ and $S_{q+1} = 0$.

By summation by parts, we have

$$\begin{aligned}\sum_{k=p}^{q}\lambda_k(v_k - v_{k-1}) &= \sum_{k=p}^{q}S_k\lambda_k - \sum_{k=p}^{q}S_{k+1}\lambda_k\\ &= \sum_{k=p}^{q}S_k\lambda_k - \sum_{k=p+1}^{q+1}S_k\lambda_{k-1}\\ &= \sum_{k=p+1}^{q}S_k(\lambda_k - \lambda_{k-1}) + S_p\lambda_p - S_{q+1}\lambda_q.\end{aligned}$$

Since (λ_n) is an increasing sequence of positive real numbers, we obtain

$$\begin{aligned}\left|\sum_{k=p}^{q}\lambda_k\Delta v_k\right| &\le \max_{p\le r\le q}\left|\sum_{k=r}^{q}\Delta v_k\right|\left(\sum_{k=p+1}^{q}|\lambda_k - \lambda_{k-1}| + |\lambda_p|\right)\\ &= \lambda_q \max_{p\le r\le q}\left|\sum_{k=r}^{q}\Delta v_k\right|. \quad \square\end{aligned}$$

Lemma 5. *Let (p_n) satisfy the condition (6). If (u_n) is slowly oscillating, then $(V_{n,p}^{(0)}(\Delta u))$ is slowly oscillating and bounded.*

Proof. Let (6) be satisfied. It is easy to see that if (u_n) is slowly oscillating, then $V_{n,p}^{(0)}(\Delta u) = O(1)$ by choosing $\lambda_k = P_{k-1}$ in Lemma 4. By the identity $V_{n,p}^{(0)}(\Delta u) = \frac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(1)}(u)$, we have $\Delta\sigma_{n,p}^{(1)}(u) = \frac{p_n}{P_{n-1}}V_{n,p}^{(0)}(\Delta u)$. Then, we have

$$\begin{aligned}&|\sigma_{n,p}^{(1)}(u) - \sigma_{n-1,p}^{(1)}(u)| \le \frac{p_n}{P_{n-1}}C,\\ &|\sigma_{k,p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u)| \le \left(\frac{p_k}{P_{k-1}} + \frac{p_{k-1}}{P_{k-2}} + \cdots + \frac{p_{n+1}}{P_n}\right)C\\ &|\sigma_{k,p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u)| \le \left(\frac{P_k - P_n}{P_n}\right)C\end{aligned}$$

for some $C > 0$. Taking max of both sides of the inequality over k, where k runs from $n+1$ to $[\lambda n]$, we obtain

$$\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_{j,p}^{(1)}(u)\right| \le \left(\frac{P_{[\lambda n]} - P_n}{P_n}\right)C.$$

Taking lim sup of both sides of the inequality above as $n\to\infty$, we have

$$\limsup_{n\to\infty}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_{j,p}^{(1)}(u)\right| \le \limsup_{n\to\infty}\left(\frac{P_{[\lambda n]} - P_n}{P_n}\right)C. \tag{9}$$

Taking the limit of both sides as $\lambda\to 1^+$, we obtain by (6)

$$\lim_{\lambda\to 1^+}\limsup_{n\to\infty}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_{j,p}^{(1)}(u)\right| \le 0. \tag{10}$$

It follows from (10) that $(\sigma_{n,p}^{(1)}(u))$ is slowly oscillating. Since (u_n) is slowly oscillating, $(V_{n,p}^{(0)}(\Delta u))$ is slowly oscillating by the weighted Kronecker identity. □

The following difference $u_n - \sigma_{n,p}^{(1)}(u)$ for a sequence $u = (u_n)$ is well known and an easy consequence of (2).

Lemma 6 ([6]). *Let (u_n) be a sequence of real numbers. For $\lambda > 1$ and sufficiently large n,*

$$u_n - \sigma_{n,p}^{(1)}(u) = \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(\sigma_{[\lambda n],p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u)\right) - \frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k(u_k - u_n)$$

where $[\lambda n]$ denotes the integer part of λn.

3. Proof of Theorem 1

By hypothesis, (u_n) is (J, p) summable to s. Since (u_n) is slowly oscillating sequence, we have from Lemma 5 that $V_{n,p}^{(0)}(\Delta u) = O(1)$. We thus obtain $V_{n,p}^{(0)}(\Delta u) \geq -C$ for some $C > 0$. From Theorem 3, (u_n) is $(\overline{N}, p)$ summable to s. It follows from the weighted Kronecker identity that $(V_{n,p}^{(1)}(\Delta u))$ is convergent to zero. Since (u_n) is slowly oscillating sequence, $(V_{n,p}^{(0)}(\Delta u))$ is slowly oscillating by Lemma 5. It suffices to show that $V_{n,p}^{(0)}(\Delta u) = o(1)$. By Lemma 6, we get

$$\begin{aligned} V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u) &= \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(V_{[\lambda n],p}^{(1)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right) - \frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k(V_{k,p}^{(0)}(\Delta u) - V_{n,p}^{(0)}(\Delta u)) \\ &= \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(V_{[\lambda n],p}^{(1)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right) - \frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k \sum_{j=n+1}^{k} \Delta V_{j,p}^{(0)}(\Delta u). \end{aligned}$$

By the triangle inequality, we have

$$\left|V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right| \leq \left|\frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(V_{[\lambda n],p}^{(1)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right)\right| + \max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_{j,p}^{(0)}(\Delta u)\right|. \tag{11}$$

Taking lim sup of both sides of the inequality (11), we obtain

$$\begin{aligned} \limsup_{n\to\infty}\left|V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right| &\leq \limsup_{n\to\infty}\frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\limsup_{n\to\infty}\left|V_{[\lambda n],p}^{(1)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right| \\ &\quad + \limsup_{n\to\infty}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_{j,p}^{(0)}(\Delta u)\right|. \end{aligned} \tag{12}$$

Since the first term on the right-hand side of the inequality (12) vanishes by (6), we have,

$$\limsup_{n\to\infty}\left|V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right| \leq \limsup_{n\to\infty}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta V_{j,p}^{(0)}(\Delta u)\right|.$$

Since $(V_{n,p}^{(0)}(\Delta u))$ is slowly oscillating, taking the limit of both sides as $\lambda \to 1^+$, we obtain $\limsup_{n\to\infty}\left|V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u)\right| \leq 0$, which implies that $V_{n,p}^{(0)}(\Delta u) = o(1)$. Since (u_n) is $(\overline{N}, p)$ summable to s, (u_n) is convergent to s by the weighted Kronecker identity. □

4. Proof of Theorem 2

By hypothesis, (u_n) is $(J, p)(\overline{N}_m, p)$ summable to s. That is, $(\sigma_{n,p}^{(m)}(u))$ is (J, p) summable to s for some nonnegative integer m. Since (u_n) is slowly oscillating, we have from Lemma 5 that $V_{n,p}^{(0)}(\Delta u) = O(1)$. From here we get $V_{n,p}^{(m)}(\Delta u) = O(1)$. We thus obtain $V_{n,p}^{(m)}(\Delta u) \geq -C$ for some $C > 0$. From Theorem 3, $(\sigma_{n,p}^{(m)}(u))$ is $(\overline{N}, p)$ summable to s. Therefore we have $(\sigma_{n,p}^{(m+1)}(u))$ is convergent to s. Since (u_n) is slowly oscillating sequence, $(\sigma_{n,p}^{(m)}(u))$ is slowly oscillating from Lemma 5. It suffices to show that $(\sigma_{n,p}^{(m)}(u))$ is convergent to s. By Lemma 6, we get

$$\begin{aligned} \sigma_{n,p}^{(m)}(u) - \sigma_{n,p}^{(m+1)}(u) &= \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(\sigma_{[\lambda n],p}^{(m+1)}(u) - \sigma_{n,p}^{(m+1)}(u)\right) - \frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k(\sigma_{k,p}^{(m)}(u) - \sigma_{n,p}^{(m)}(u)) \\ &= \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(\sigma_{[\lambda n],p}^{(m+1)}(u) - \sigma_{n,p}^{(m+1)}(u)\right) - \frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k \sum_{j=n+1}^{k} \Delta\sigma_{j,p}^{(m)}(\Delta u). \end{aligned}$$

By the triangle inequality, we have

$$\left|\sigma_{n,p}^{(m)}(u)-\sigma_{n,p}^{(m+1)}(u)\right| \leq \left|\frac{P_{[\lambda n]}}{P_{[\lambda n]}-P_n}\left(\sigma_{[\lambda n],p}^{(m+1)}(u)-\sigma_{n,p}^{(m+1)}(u)\right)\right| + \max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_{j,p}^{(m)}(u)\right|. \tag{13}$$

Taking lim sup of both sides of the inequality (13), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\left|\sigma_{n,p}^{(m)}(u)-\sigma_{n,p}^{(m+1)}(u)\right| \leq{}& \limsup_{n\to\infty}\frac{P_{[\lambda n]}}{P_{[\lambda n]}-P_n}\limsup_{n\to\infty}\left|\sigma_{[\lambda n],p}^{(m+1)}(u)-\sigma_{n,p}^{(m+1)}(u)\right| \\ &+ \limsup_{n\to\infty}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_{j,p}^{(m)}(u)\right|. \end{aligned}\tag{14}$$

Since the first term on the right-hand side of the inequality (14) vanishes by (6), we have

$$\limsup_{n\to\infty}\left|\sigma_{n,p}^{(m)}(u)-\sigma_{n,p}^{(m+1)}(u)\right| \leq \limsup_{n\to\infty}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_{j,p}^{(m)}(u)\right|.$$

Since $(\sigma_{n,p}^{(m)}(u))$ is slowly oscillating, taking the limit of both sides as $\lambda\to 1^+$, we obtain $\limsup_{n\to\infty}\left|\sigma_{n,p}^{(m)}(u)-\sigma_{n,p}^{(m+1)}(u)\right| \leq 0$, which implies that $(\sigma_{n,p}^{(m)}(u))$ is convergent to s. This means that $(\sigma_{n,p}^{(m-1)}(u))$ is $(\overline{N}, p)$ summable to s. Therefore, we have $(\sigma_{n,p}^{(m-1)}(u))$ is (J, p) summable to s. Repeating the above process $m-1$ times we obtain (u_n) is (J, p) summable to s. This completes the proof by Theorem 1. □

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